WEIERSTRASS FILTRATION ON TEICHMÜLLER CURVES AND LYAPUNOV EXPONENTS: UPPER BOUNDS

FEI YU AND KANG ZUO

ABSTRACT. We get an upper bound of the slope of each graded quotient for the Harder-Narasimhan filtration of the Hodge bundle of a Teichmüller curve. As an application, we show that the sum of Lyapunov exponents of a Teichmüller curve does not exceed $(g+1)/2$, with equality reached if and only if the curve lies in the hyperelliptic locus induced from $\mathcal{Q}(2k_1,...,2k_n,-1^{2g+2})$ or it is some special Teichmüller curve in $\Omega\mathcal{M}_g(1^{2g-2})$. Under some additional assumptions, we also get an upper bound of individual Lyapunov exponents; in particular we get Lyapunov exponents in hyperelliptic loci and low genus non-varying strata.

CONTENTS

1. INTRODUCTION

Let $\mathcal{M}_g$ be the moduli space of Riemann surfaces of genus g, and $\Omega\mathcal{M}_g \to \mathcal{M}_g$ the bundle of pairs (X,ω), where $\omega \neq 0$ is a holomorphic 1-form on $X \in \mathcal{M}_g$. Denote $\Omega\mathcal{M}_g(m_1,...m_k) \hookrightarrow \Omega\mathcal{M}_g$ the stratum of pairs (X,ω), where $\omega(\neq 0)$ have k distinct zeros of order $m_1,...,m_k$ respectively.

There is a nature action of $GL_2^+(\mathbb{R})$ on $\Omega\mathcal{M}_g(m_1,...m_k)$, whose orbits project to complex geodesics in $\mathcal{M}_g$. The projection of an orbit is almost always dense. However, if the stabilizer $SL(X,\omega) \subset SL_2(\mathbb{R})$ of a given form is a lattice, then the projection of its orbit gives a closed, algebraic Teichmüller curve C.

After suitable base change and compacfication, we can get a universal family $f: S \to C$, which is a relative minimal semistable model with disjoint sections $D_1,..,D_k$; here $D_i|_X$ is a zero of ω when restrict to each fiber X.

Date: September 25, 2012.
This work is supported by the SFB/TR 45 Periods, Moduli Spaces and Arithmetic of Algebraic Varieties of the DFG.

The relative canonical bundle formula (1) of the Teichmüller curve is([4][8]):

$$\omega_{S/C} \simeq f^*\mathcal{L} \otimes \mathcal{O}(\sum_i m_i D_i)$$

Here $\mathcal{L} \subset f_*\omega_{S/C}$ be the line bundle whose fiber over the point corresponding to X is $\mathbb{C}\omega$, the generating differential of Teichmüller curves.

There are many nature vector subbundles of the Hodge bundle $f_*(\omega_{S/C})$:

$$\mathcal{L} \otimes f_*\mathcal{O}(\sum d_i D_i) \subset \mathcal{L} \otimes f_*\mathcal{O}(\sum m_i D_i) = f_*(\omega_{S/C})$$

One can construct many filtration by using these subbundles. In particular, using properties of Weierstrass semigroups, we have constructed the Harder-Narasimhan filtration of $f_*(\omega_{S/C})$ for Teichmüller curves in hyperelliptic loci and some low genus nonvarying strata [23]. In this article, we will get an upper bound of the slope of each graded quotient for the Harder-Narasimhan filtration of $f_*(\omega_{S/C})$ of Teichmüller curves in each stratum.

For a vector bundle V, define $\mu_i(V) = \mu(gr_j^{HN})$ if $rk(HN_{j-1}(V)) < i \leq rk(HN_j(V))$. Write w_i for $\mu_i(f_*(\omega_{S/C}))/deg(\mathcal{L})$.

Lemma 1.1. *(Lemma 5.4) For a Teichmüller curve which lies in $\Omega\mathcal{M}_g(m_1, ...m_k)$, we have inequalities:*

$$w_i \leq 1 + a_{H_i(P)}$$

Here a_i is the i-th largest number in $\{-\frac{j}{m_i+1} | 1 \leq j \leq m_i, 1 \leq i \leq k\}$, P is the special permutation (4) and $H_i(P) \geq 2i - 2$.

Fix an $SL_2(\mathbb{R})$-invariant, ergodic measure μ on $\Omega\mathcal{M}_g$. The Lyapunov exponents for the Teichmüller geodesic flow on $\Omega\mathcal{M}_g$ measure the logarithm of the growth rate of the Hodge norm of cohomology classes under the parallel transport along the geodesic flow.

In general, it is difficult to compute the Lyapunov exponents. There are some algebraic attempts to compute the sum of certain Lyapunov exponents, all of which are based on the following fact: the sum of these Lyapunov exponents is related with the degree of certain vector bundles (cf. Theorem 4.1). In particular, the sum of Lyapunov exponents of a Teichmüller curve equals $deg(f_*(\omega_{S/C}))/deg(\mathcal{L})$. This algebraic interpretation combined with information about the Harder-Narasimhan filtration gives the following estimate:

Theorem 1.2. *(Theorem 5.2) The sum of Lyapunov exponents of a Teichmüller curve in $\Omega\mathcal{M}_g(m_1, ...m_k)$ satisfies the inequality*

$$L(C) \leq \frac{g+1}{2}$$

Furthermore, equality occurs if and only if it lies in the hyperelliptic locus induced from $\mathcal{Q}(2k_1, ..., 2k_n, -1^{2g+2})$ or it is some special Teichmüller curve in $\Omega\mathcal{M}_g(1^{2g-2})$.

D.W. Chen and M. Möller have obtained many interesting upper bounds in [4][20].

The Harder-Narasimhan filtration also gives rise to an upper bound of the degrees of any vector subbundles, especially those related to the sum of certain Lyapunov exponents (cf. Proposition 5.5).

For individual Lyapunov exponents, due to the lack of algebraic interpretation, we will make the following assumption:

Assumption 1.3. *$f_*(\omega_{S/C})$ equals $(\bigoplus_{i=1}^{k} L_i) \oplus W$, here L_i are line bundles such that the i-th Lyapunov exponent satisfies the equality:*

$$\lambda_i = \left\{ \begin{array}{ll} deg(L_i)/deg(\mathcal{L}) & 1 \leq i \leq k \\ 0 & k < i \leq g \end{array} \right.$$

There are many examples satisfying this assumption: triangle groups [2], square tiled cyclic covers [7][11], square tiled abelian covers [22], some wind-tree models [6], and algebraic primitives.

Our estimate on the slopes of the Harder-Narasimhan filtration will give the following upper bound for individual Lyapunov exponents:

Proposition 1.4. *(Proposition 5.7) For a Teichmüller curve which satisfies the assumption 1.3 and lies in $\Omega\mathcal{M}_g(m_1, ...m_k)$, the i-th Lyapunov exponent satisfies the inequality:*

$$\lambda_i \leq 1 + a_{H_i(P)}$$

Here a_i is the i-th largest number in $\{-\frac{j}{m_i+1} | 1 \leq j \leq m_i, 1 \leq i \leq k\}$, P is the special permutation (4) *and $H_i(P) \geq 2i - 2$.*

The equality can be reached for an algebraic primitive Teichmüller curve lying in the hyperelliptic locus induced from $\mathcal{Q}(2k_1, ..., 2k_n, -1^{2g+2})$.

For Teichmüller curves lying in hyperelliptic loci and some low genus nonvarying strata, the following proposition is obvious because we have constructed the Harder-Narasimhan filtration in [23].

Proposition 1.5. *(Proposition 7.1) For a Teichmüller curve which satisfies the assumption 1.3 and lies in hyperelliptic loci or one of the following strata:*
$\overline{\Omega\mathcal{M}}_3(4), \overline{\Omega\mathcal{M}}_3(3,1), \overline{\Omega\mathcal{M}}_3^{odd}(2,2), \overline{\Omega\mathcal{M}}_3(2,1,1)$
$\overline{\Omega\mathcal{M}}_4(6), \overline{\Omega\mathcal{M}}_4(5,1), \overline{\Omega\mathcal{M}}_4^{odd}(4,2), \overline{\Omega\mathcal{M}}_4^{non-hyp}(3,3), \overline{\Omega\mathcal{M}}_4^{odd}(2,2,2), \overline{\Omega\mathcal{M}}_4(3,2,1)$
$\overline{\Omega\mathcal{M}}_5(8), \overline{\Omega\mathcal{M}}_5(5,3), \overline{\Omega\mathcal{M}}_5^{odd}(6,2)$
The i-th Lyapunov exponent λ_i equals the w_i which is computed in the theorem 3.5.

Acknowledgement. We thank Ke Chen for a careful reading and his many suggestions.

2. Harder-Narasimhan filtration

The readers are referred to [12] for details about sheaves on algebraic varieties. Let C be a smooth projective curve, V a vector bundle over C of slope $\mu(V) := \frac{deg(V)}{rk(V)}$. We call V semistable (resp.stable) if $\mu(W) \leq \mu(V)$ (resp.$\mu(W) < \mu(V)$) for any subbundle $W \subset V$. If V_1, V_2 are semistable such that $\mu(V_1) > \mu(V_2)$, then any map $V_1 \to V_2$ is zero.

A Harder-Narasimhan filtration for V is an increasing filtration:

$$0 = HN_0(V) \subset HN_1(V) \subset ... \subset HN_k(V)$$

such that the graded quotients $gr_i^{HN} = HN_i(V)/HN_{i-1}(V)$ for $i = 1, ..., k$ are semistable vector bundles and

$$\mu(gr_1^{HN}) > \mu(gr_2^{HN}) > ... > \mu(gr_k^{HN})$$

The Harder-Narasimhan filtration is unique.

A Jordan-Hölder filtration for semistable vector bundle V is a filtration:

$$0 = V_0 \subset V_1 \subset ... \subset V_k = V$$

such that the graded quotients $gr_i^V = V_i/V_{i-1}$ are stable of the same slope.

Jordan-Hölder filtration always exist. The graded objects $gr_i^V = \oplus gr_i^V$ do not depend on the choice of the Jordan-Hölder filtration.

For a vector bundle V, define $\mu_i(V) = \mu(gr_j^{HN})$ if $rk(HN_{j-1}(V)) < i \leq rk(HN_j(V))$. Obviously we have $\mu_1(V) \geq ... \geq \mu_k(V)$.

Lemma 2.1. *Let V and U be two vector bundles of rank n over C, with increasing filtration*

$$0 = V_0 \subset V_1 \subset ... \subset V_n = V$$

$$0 = U_0 \subset U_1 \subset ... \subset U_n = U$$

such that $V_i/V_{i-1}, U_i/U_{i-1}$ are line bundles , $V_i/V_{i-1} \subset U_i/U_{i-1}$ and the degrees $deg(U_i/U_{i-1})$ decrease in i ($1 \leq i \leq n$). Then $\mu_i(V) \leq deg(U_i/U_{i-1})$.

Proof. If there is some $\mu_i(V)$ bigger than $deg(U_i/U_{i-1})$, where $\mu_i(V) = \mu(gr_j^{HN(V)})$, then $\mu_i(V) > deg(U_i/U_{i-1}) \geq deg(U_l/U_{l-1}) \geq deg(V_l/V_{l-1})$, for $l \geq i$.

We will show that the canonical morphism $HN_j(V) \hookrightarrow V \to V/V_{i-1}$ is zero, namely $HN_j(V) \hookrightarrow V_{i-1}$, which is a contradiction because $rk(HN_j(V)) \geq i > rk(V_{i-1})$.

For $m \leq j, l \geq i$, the quotients $gr_m^{HN(V)}, V_l/V_{l-1}$ are semistable and $\mu(gr_m^{HN(V)}) \geq \mu_i(V) > deg(V_l/V_{l-1})$, so any map $gr_m^{HN(V)} \to V_l/V_{l-1}$ is zero. Thus any map $gr_m^{HN(V)} \to V/V_{i-1}$ is zero by induction on l, and any map $HN_j(V) \to V/V_{i-1}$ is zero by induction on m. □

Let $grad(HN(V))$ denote the direct sum of the graded quotients of the Harder-Narasimhan filtration: $grad(HN(V)) = \oplus gr_i^{HN(V)}$.

Lemma 2.2. *Given vector bundles $V_1, ..., V_n$, we have:*

$$grad(HN(V_1 \oplus ... \oplus V_n)) = grad(HN(V_1)) \oplus ... \oplus grad(HN(V_n))$$

and $\mu_i(V_j)$ equals $\mu_k(V_1 \oplus ... \oplus V_n)$ for some k.

Proof. By induction, we only need to show the case $n = 2$. Let

$$0 = HN_0(V_1) \subset HN_1(V_1) \subset ... \subset HN_{k_1}(V_1)$$

$$0 = HN_0(V_2) \subset HN_1(V_2) \subset ... \subset HN_{k_2}(V_2)$$

be the Harder-Narasimhan filtration of V_1, V_2 respectively.

Set $0 = HN_0(V_1 \oplus V_2) = HN_0(V_1) \oplus HN_0(V_2)$. Assume we have set $HN_i(V_1 \oplus V_2) = HN_{i_1}(V_1) \oplus HN_{i_2}(V_2)$. We will get $HN_{i+1}(V_1 \oplus V_2)$ by the following rule:

- If $\mu(HN_{i_1+1}(V_1)/HN_{i_1}(V_1)) > \mu(HN_{i_2+1}(V_2)/HN_{i_2}(V_2))$ then let $HN_{i+1}(V_1 \oplus V_2) = HN_{i_1+1}(V_1) \oplus HN_{i_2}(V_2)$.
- If $\mu(HN_{i_1+1}(V_1)/HN_{i_1}(V_1)) = \mu(HN_{i_2+1}(V_2)/HN_{i_2}(V_2))$ then let $HN_{i+1}(V_1 \oplus V_2) = HN_{i_1+1}(V_1) \oplus HN_{i_2+1}(V_2)$.
- If $\mu(HN_{i_1+1}(V_1)/HN_{i_1}(V_1)) < \mu(HN_{i_2+1}(V_2)/HN_{i_2}(V_2))$ then let $HN_{i+1}(V_1 \oplus V_2) = HN_{i_1}(V_1) \oplus HN_{i_2+1}(V_2)$.

It is easy to check that the vector bundle $gr_i^{HN(V_1 \oplus V_2)} = HN_{i+1}(V_1 \oplus V_2)/HN_i(V_1 \oplus V_2)$ is semistable of slope

$$max\{\mu(gr_{i_1+1}^{HN(V_1)}), \mu(gr_{i_2+1}^{HN(V_2)})\}$$

and the slope is strictly decreasing in i. We have thus constructed the Harder-Narasimhan filtration of $V_1 \oplus V_2$. From the construction, we also have

$$grad(HN(V_1 \oplus V_2)) = grad(HN(V_1)) \oplus grad(HN(V_2))$$

and $\mu_i(V_1) = \mu(gr_j^{HN(V_1)})$ always equals $\mu_k(V_1 \oplus V_2)$ for some k.

□

3. Filtration of the Hodge bundle

Let $\Omega\mathcal{M}_g(m_1, ..., m_k)$ be the stratum parameterizing (X, ω) where X is a curve of genus g and ω is an Abelian differentials (i.e.a holomorphic one-form) on X that have k distinct zeros of order $m_1, ..., m_k$. Denote by $\overline{\Omega\mathcal{M}}_g(m_1, ..., m_k)$ the Deligne-Mumford compactification of $\Omega\mathcal{M}_g(m_1, ..., m_k)$. Denote by $\Omega\mathcal{M}_g^{hyp}(m_1, ..., m_k)$(resp. odd, resp. even) the hyperelliptic (resp. odd theta character, resp. even theta character) connected component. ([15])

Let $\mathcal{Q}(d_1, ..., d_n)$ be the stratum parameterizing (X, q) where X is a curve of genus g and q is a meromorphic quadratic differentials with at most simple zeros on X that have k distinct zeros of order $d_1, ..., d_n$ respectively.

If the quadratic differential is not a global square of a one-form, there is a canonical double covering $\pi : Y \to X$ such that $\pi^* q = \omega^2$. This covering is ramified precisely at the zeros of odd order of q and at the poles. It give a map

$$\phi : \mathcal{Q}(d_1, ..., d_n) \to \Omega\mathcal{M}_g(m_1, ..., m_k)$$

A singularity of order d_i of q give rise to two zeros of degree $m = d_i/2$ when d_i is even, single zero of degree $m = d + 1$ when d is odd. Especially, the hyperelliptic locus in a stratum $\Omega\mathcal{M}_g(m_1, ..., m_k)$ induces from a stratum $\mathcal{Q}(d_1, ..., d_k)$ satisfying $d_1 + ... + d_n = -4$.

There is a nature action of $GL_2^+(\mathbb{R})$ on $\Omega\mathcal{M}_g(m_1, ..., m_k)$, whose orbits project to complex geodesics in $\mathcal{M}_g$. The projection of an orbit is almost always dense. If the stabilizer $SL(X, \omega) \subset SL_2(\mathbb{R})$ of given form is a lattice, however, then the projection of its orbit gives a closed, algebraic Teichmüller curve C.

The Teichmüller curve C is an algebraic curve in $\overline{\Omega\mathcal{M}}_g$ that is totally geodesic with respect to the Teichmüller metric.

After suitable base change, we can get a universal family $f : S \to C$, which is a relatively minimal semistable model with disjoint sections $D_1, .., D_k$; here $D_i|_X$ is a zero of ω when restrict to each fiber X. ([4])

Let $\mathcal{L} \subset f_*\omega_{S/C}$ be the line bundle whose fiber over the point corresponding to X is $\mathbb{C}\omega$, the generating differential of Teichmüller curves; it is also known as the "maximal Higgs" line bundle. Let $\Delta \subset \overline{B}$ be the set of points with singular fibers, then the property of being "maximal Higgs" says by definition that $\mathcal{L} \cong \mathcal{L}^{-1} \otimes \omega_C(log\Delta)$ and

$$deg(\mathcal{L}) = (2g(C) - 2 + |\Delta|)/2,$$

together with an identification (relative canonical bundle formula)([4][8]):

$$\omega_{S/C} \simeq f^*\mathcal{L} \otimes \mathcal{O}(\Sigma m_i D_i) \tag{1}$$

By the adjunction formula we get

$$D_i^2 = -\omega_{S/C} D_i = -m_i D_i^2 - deg\mathcal{L}$$

and thus

$$D_i^2 = -\frac{1}{m_i+1} deg\mathcal{L} \tag{2}$$

For a line bundle $\mathcal{L}$ of degree d on X, denote by $h^0(\mathcal{L})$ the dimension of $dim(H^0(X,\mathcal{L}))$. From the exact sequence

$$0 \to f_*\mathcal{O}(d_1D_1 + ... + d_kD_k) \to f_*\mathcal{O}(m_1D_1 + ... + m_kD_k) = f_*(\omega_{S/C}) \otimes \mathcal{L}^{-1}$$

and the fact that all subsheaves of a locally free sheaf on a curve are locally free, we deduce that $f_*\mathcal{O}(d_1D_1 + ... + d_kD_k)$ is a vector bundle of rank $h^0(d_1p_1 + ... + d_kp_k)$, here $p_i = D_i|_F$, F is a generic fiber. We have constructed many filtration of the Hodge bundle by using those vector bundles in [23].

A fundament exact sequence for those filtration is the following:

$$0 \to f_*\mathcal{O}(\sum(d_i - a_i)D_i) \to f_*\mathcal{O}(\sum d_iD_i) \to f_*\mathcal{O}_{\sum a_iD_i}(\sum d_iD_i) \xrightarrow{\delta} \tag{3}$$

$$R^1f_*\mathcal{O}(\sum(d_i - a_i)D_i) \to R^1f_*\mathcal{O}(\sum d_iD_i) \to 0$$

There are many properties of these filtration:

Lemma 3.1. *([23]) If $h^0(\sum d_ip_i) = h^0(\sum(d_i - a_i)p_i)$ holds in a general fiber, then we have the equality $f_*\mathcal{O}(\sum d_iD_i) = f_*\mathcal{O}(\sum(d_i - a_i)D_i)$.*

Lemma 3.2. *([23]) If $h^0(\sum d_ip_i) = h^0(\sum(d_i - a_i)p_i) + \sum a_i$ is non-varying, then*

$$f_*\mathcal{O}(\sum d_iD_i)/f_*\mathcal{O}(\sum(d_i - a_i)D_i) = f_*\mathcal{O}_{\sum a_iD_i}(\sum d_iD_i) = \oplus f_*\mathcal{O}_{a_iD_i}(d_iD_i)$$

Lemma 3.3. *([23]) The Harder-Narasimhan filtration of $f_*\mathcal{O}_{aD}(dD)$ is*

$$0 \subset f_*\mathcal{O}_D((d-a+1)D)... \subset ...f_*\mathcal{O}_{(a-1)D}((d-1)D) \subset f_*\mathcal{O}_{aD}(dD)$$

and the direct sum of the graded quotient of this filtration is

$$grad(HN(f_*\mathcal{O}_{aD}(dD))) = \bigoplus_{i=0}^{a-1} \mathcal{O}_D((d-i)D)$$

Lemma 3.4. *([23])The degree $deg(f_*\mathcal{O}(\sum d_iD_i)/f_*\mathcal{O}(\sum(d_i - a_i)D_i))$ is smaller than the maximal sums of $h^0(\sum d_ip_i) - h^0(\sum(d_i - a_i)p_i)$ line bundles in*

$$\bigcup_i \bigcup_{j=0}^{a_i-1} \mathcal{O}_{D_i}((d_i - j)D_i)$$

(here $p_i = D_i|_F$, F being a general fiber).

For a Teichmüller curves lying in hyperelliptic loci and low genus non-varying strata, we have constructed the Harder-Narasimhan filtration.

Write w_i for $\mu_i(f_*(\omega_{S/C}))/deg(\mathcal{L})$.

Theorem 3.5. [23] *Let C be a Teichmüller curve in the hyperelliptic locus of a stratum $\overline{\Omega\mathcal{M}}_g(m_1, ..., m_k)$, and denote by $(d_1, ...d_n)$ the orders of singularities of underlying quadratic differentials. Then the w_i's for C are*

$$1, \{1 - \frac{2k}{d_j+2}\}_{0<2k\leq d_j+1}$$

Table 1. genus 3

zeros	component	w_2	w_3	$\sum w_i$
(4)	hyp	3/5	1/5	9/5
(4)	odd	2/5	1/5	8/5
(3,1)		2/4	1/4	7/4
(2,2)	hyp	2/3	1/3	2
(2,2)	odd	1/3	1/3	5/3
(2,1,1)		1/2	1/3	11/6
(1,1,1,1)				≤ 2

Table 2. genus 4

zeros	component	w_2	w_3	w_4	$\sum w_i$
(6)	hyp	5/7	3/7	1/7	16/7
(6)	even	4/7	2/7	1/7	14/7
(6)	odd	3/7	2/7	1/7	13/7
(5,1)		1/2	2/6	1/6	2
(3,3)	hyp	3/4	2/4	1/4	5/2
(3,3)	non-hyp	2/4	1/4	1/4	2
(4,2)	even	3/5	1/3	1/5	32/15
(4,2)	odd	2/5	1/3	1/5	29/15
(2,2,2)		1/3	1/3	1/3	2
(3,2,1)		1/2	1/3	1/4	25/12

Table 3. genus 5

zeros	component	w_2	w_3	w_4	w_5	$\sum w_i$
(8)	hyp	7/9	5/9	3/9	1/9	25/9
(8)	even	5/9	3/9	2/9	1/9	20/9
(8)	odd	4/9	3/9	2/9	1/9	19/9
(5,3)		1/2	1/3	1/4	1/6	9/4
(6,2)	odd	3/7	1/3	2/7	1/7	46/21
(4,4)	hyp	4/5	3/5	2/5	1/5	3

For a Teichmüller curve lying in some low genus non varying strata, the w_i's are computed in Table 1, Table 2, Table 3.

4. Lyapunov exponents

A good introduction to Lyapunov exponents with a lot of motivating examples is the survey by Zorich ([24]).

Fix an $SL_2(\mathbb{R})$-invariant, ergodic measure μ on $\Omega\mathcal{M}_g$. Let V be the restriction of the real Hodge bundle (i.e. the bundle with fibers $H^1(X,\mathbb{R})$) to the support M of μ. Let S_t be the lift of the geodesic flow to V via the Gauss-Manin connection. Then Oseledec's multiplicative ergodic theorem guarantees the existence of a filtration

$$0 \subset V_{\lambda_g} \subset ... \subset V_{\lambda_1} = V$$

by measurable vector subbundles with the property that, for almost all $m \in M$ and all $v \in V_m \backslash \{0\}$ one has

$$||S_t(v)|| = exp(\lambda_i t + o(t))$$

where i is the maximal index such that v is in the fiber of V_i over m i.e.$v \in (V_i)_m$. The numbers λ_i for $i = 1, ..., k \leq rank(V)$ are called the *Lyapunov exponents* of S_t. Since V is symplectic, the spectrum is symmetric in the sense that $\lambda_{g+k} = -\lambda_{g-k+1}$. Moreover, from elementary geometric arguments it follows that one always has $\lambda_1 = 1$.

There is an algebraic interpretation of the sum of certain Lyapunov exponents:

Theorem 4.1. *(*[14][10][2]*)If the Variation of Hodge structure (VHS) over the Teichmüller curve C contains a sub-VHS $\mathbb{W}$ of rank $2k$, then the sum of the k corresponding to non-negative Lyapunov exponents equals*

$$\sum_{i=1}^{k} \lambda_i^{\mathbb{W}} = \frac{2deg\mathbb{W}^{(1,0)}}{2g(C) - 2 + |\Delta|}$$

where $\mathbb{W}^{(1,0)}$ is the $(1,0)$-part of the Hodge filtration of the vector bundle associated with $\mathbb{W}$. In particular, we have

$$\sum_{i=1}^{g} \lambda_i = \frac{2deg f_* \omega_{S/C}}{2g(C) - 2 + |\Delta|}$$

Let $L(C) = \sum\limits_{i=1}^{g} \lambda_i$ be the sum of Lyapunov exponents, and put $k_\mu = \frac{1}{12} \sum\limits_{i=1}^{k} \frac{m_i(m_i+2)}{m_i+1}$. Eskin, Kontsevich and Zorich obtain a formula to compute $L(C)$ (for the Teichmüller geodesic flow):

Theorem 4.2. *(*[8]*)For the VHS over the Teichmüller curve C, we have*

$$L(C) = k_\mu + \frac{\pi^2}{3} c_{area}(C)$$

where $c_{area}(C)$ is the Siegel-Veech constant corresponding to C.

Because the Siegel-Veech constant is non-negative, there is a lower bound $L(C) \geq k_\mu$.

5. Upper bounds

Denote by $|\mathcal{L}|$ the projective space of one-dimensional subspaces of $H^0(X, \mathcal{L})$. For a (projective) r-dimension linear subspace V of $|\mathcal{L}|$, we call $(\mathcal{L}, V)$ a linear series of type g_d^r.

Theorem 5.1 (Clifford's theorem [13])**.** *Let $\mathcal{L}$ be an effective special divisor (i.e. $h^1(\mathcal{L}) \neq 0$) on the curve X. Then*

$$h^0(\mathcal{L}) \leq 1 + \frac{1}{2} deg(\mathcal{L})$$

Furthermore, equality occurs if and only if either $\mathcal{L} = 0$ or $\mathcal{L} = K$ or X is hyperelliptic and $\mathcal{L}$ is a multiple of the unique linear series of type g_2^1 on X.

Let C be a Teichmüller curve lying in $\Omega\mathcal{M}_g(m_1,...m_k)$. Let $P=(p'_1,...,p'_{2g-2})$ be a permutation of $2g-2$ points

$$\overbrace{\underbrace{p_1,...,p_1}_{m_1},...,\underbrace{p_k,...,p_k}_{m_k}}^{2g-2}$$

The point p_i is the intersection of the section D_i with the generic fiber F.

For $j=1,...,g$, denote $H_j(P)=i$ if $h^0(p'_1+...+p'_{i-1})=j-1$ and $h^0(p'_1+...+p'_i)=j$.

First by Clifford's Theorem $h^0(p'_1+...+p'_i)\le 1+\frac{deg(p'_1+...+p'_i)}{2}$, we have $H_j(P)\ge 2j-2$. When $j<g$, if the equality holds then C lies in the hyperelliptic locus.

Next by using vector bundles $f_*\mathcal{O}(D'_1+...+D'_i),(1\le i\le 2g-2)$, we construct a filtration

$$0\subset V'_1\subset V'_2...\subset V'_g=f_*\mathcal{O}(m_1D_1+...+m_kD_k)$$

where V'_j is a rank j vector bundle and $V'_j=f_*\mathcal{O}(D'_1+...+D'_{H_j(P)})=...=f_*\mathcal{O}(D'_1+...+D'_{H_{j+1}(P)-1})$ by lemma 3.1.

From the exact sequence

$$0\to f_*\mathcal{O}(D'_1+...+D'_{H_j(P)-1})\to f_*\mathcal{O}(D'_1+...+D'_{H_j(P)})\to \mathcal{O}_{D'_{H_j(P)}}(D'_1+...+D'_{H_j(P)})$$

we see that the graded quotients V'_j/V'_{j-1} has an upper bound $\mathcal{O}_{D'_{H_j(P)}}(D'_1+...+D'_{H_j(P)})$ by lemma 3.4.

Theorem 5.2. *The sum of Lyapunov exponents of a Teichmüller curve in* $\Omega\mathcal{M}_g(m_1,...m_k)$ *satisfies the inequality*

$$L(C)\le\frac{g+1}{2}$$

Furthermore, equality occurs if and only if it lies in the hyperelliptic locus induced from $\mathcal{Q}(2k_1,...,2k_n,-1^{2g+2})$ *or it is some special Teichmüller curve in* $\Omega\mathcal{M}_g(1^{2g-2})$.

Proof. In $\Omega\mathcal{M}_g(m_1,...m_k)$, there is a direct sum decomposition ([18]):

$$f_*\omega_{S/C}=\mathcal{L}\otimes(\mathcal{O}_C\oplus f_*\mathcal{O}(m_1D_1+...+m_kD_k)/\mathcal{O}_C)$$

We want to estimate the maximal degree of the rank $g-1$ subbundle $f_*\mathcal{O}(m_1D_1+...+m_kD_k)/\mathcal{O}_C)$ because we want to obtain an upper bound of $L(C)=deg(f_*\omega_{S/C})/deg(\mathcal{L})$.

By exact sequence (3), we have

$$f_*\mathcal{O}(m_1D_1+...+m_kD_k)/\mathcal{O}_C\subset f_*\mathcal{O}_{\sum m_iD_i}(\sum m_iD_i)=\bigoplus_i f_*\mathcal{O}_{m_iD_i}(m_iD_i)$$

The last equality follows from the fact that the D_i's are disjoint. By lemma 3.3, we have $grad(HN(\mathcal{O}_{m_iD_i}(m_iD_i))=\bigoplus_{j=1}^{m_i}\mathcal{O}_{D_i}(jD_i)$. By lemma 2.2, the direct sum of the graded quotients of $HN(\bigoplus_i f_*\mathcal{O}_{m_iD_i}(m_iD_i))$ is

$$grad(HN(\bigoplus_i f_*\mathcal{O}_{m_iD_i}(m_iD_i)))=\bigoplus_i\bigoplus_{j=1}^{m_i}\mathcal{O}_{D_i}(jD_i)$$

Consider the degree of each summand, we can easily construct a filtration of $\bigoplus_i\bigoplus_{j=1}^{m_i}\mathcal{O}_{D_i}(jD_i)$:

$$0\subset V_1\subset V_2...\subset V_{2g-2}=\bigoplus_i\bigoplus_{j=1}^{m_i}\mathcal{O}_{D_i}(jD_i)$$

satisfying: 1). V_i/V_{i-1} is a line bundle, 2). $deg(V_i/V_{i-1})$ decreases in i.

We rearrange the $2g-2$ points $m_1p_1, ..., m_kp_k$ of generic fiber. If $V_i/V_{i-1} = \mathcal{O}_{D_j}(dD_j)$, then let $p'_i = p_j$. Thus we get a special permutation

$$P = (p'_1, p'_2, ..., p'_{2g-2}) \tag{4}$$

Because $D_j^2 < 0$, and $deg(\mathcal{O}_{D_j}(D_j) > ... > deg(\mathcal{O}_{D_j}((d-1)D_j) > deg(\mathcal{O}_{D_j}(dD_j)$, there are only $d-1$ p_j's appearing before p'_i

$$D'_i = D_j, D'_1 + ... + D'_i = dD_j + (\texttt{not contain } D_i \texttt{ part})$$

So

$$grad(HN(\mathcal{O}_{\sum_{k=1}^{i} D'_k}(\sum_{k=1}^{i} D'_k)))/grad(HN(\mathcal{O}_{\sum_{k=1}^{i-1} D'_k}(\sum_{k=1}^{i-1} D'_k))) = \mathcal{O}_{D_j}(dD_j)$$

By induction we get:

$$V_i = grad(HN(\mathcal{O}_{D'_1+...+D'_i}(D'_1 + ... + D'_i))) \tag{5}$$

Use vector bundles $f_*\mathcal{O}(D'_1 + ... + D'_i)$, we also construct a filtration

$$0 \subset V'_1 \subset V'_2 ... \subset V'_g = f_*\mathcal{O}(m_1D_1 + ... + m_kD_k) \tag{6}$$

where the equalities $V'_j = f_*\mathcal{O}(D'_1+...+D'_{H_j(P)}) = ... = f_*\mathcal{O}(D'_1+...+D'_{H_{j+1}(P)-1})$, by lemma 3.1.

We get the following exact sequence by using (3):

$$0 \to f_*\mathcal{O}(D'_1 + ... + D'_{H_j(P)-1}) \to f_*\mathcal{O}(D'_1 + ... + D'_{H_j(P)}) \to V_{H_j(P)}/V_{H_j(P)-1}$$

The lemma 3.4 and the Clifford theorem give us:

$$deg(V'_j/V'_{j-1}) \leq deg(V_{H_j(P)}/V_{H_j(P)-1}) \leq deg(V_{2j-2}/V_{2j-3})$$

We set $a_i := deg(V_i/V_{i-1})/deg(\mathcal{L}), b_i := deg(V'_i/V'_{i-1})/deg(\mathcal{L})$. By definition $b_1 = 0$ and a_i is the i-th largest number of $\{-\frac{j}{m_i+1} | 1 \leq j \leq m_i, 1 \leq i \leq k\}$.

Hence

$$b_j = deg(V'_j/V'_{j-1})/deg(\mathcal{L}) \leq deg(V_{2j-2}/V_{2j-3})/deg(\mathcal{L}) = a_{2j-2}$$

After some element computations:

$$\begin{aligned}\sum_{j=2}^{g} b_j &\leq \sum_{j=1}^{g-1} a_{2j} \leq \sum_{j=1}^{g-1}(a_{2j-1} + a_{2j})/2 = \frac{1}{2}\sum_{j=1}^{2g-2} a_j \\ &= \frac{1}{2}deg(\bigoplus_i \bigoplus_{j=1}^{m_i} \mathcal{O}_{D_i}(jD_i))/deg(\mathcal{L}) = \frac{1}{2}\sum_{l=1}^{k}\sum_{i=1}^{m_k}(-\frac{i}{m_l+1}) \\ &= \frac{1}{4}\sum_{l=1}^{k}(-m_l) = -\frac{g-1}{2}\end{aligned}$$

We get

$$L(C) = g + \frac{deg(f_*\mathcal{O}(m_1D_1 + ... + m_kD_k))}{deg(\mathcal{L})} \leq g + \sum_{j=1}^{g-1} b_j = g + \sum_{j=2}^{g-1} b_j = \frac{g+1}{2}$$

When the inequality becomes equal, we have $a_{2j-1} = a_{2j} = b_{j+1}$. If $b_{k+1} = a_1 = a_2 = ... = a_{2k} > a_{2k+1} = a_{2k+2} = b_{k+2}$, then the exact sequence

$$0 \to f_*\mathcal{O}(D'_1 + ... + D'_{2k}) \to f_*\mathcal{O}(D'_1 + ... + D'_{2k+1}) \to V_{2k+1}/V_{2k}$$

give us $h^0(p'_1+...+p'_{2k}) \geq k+1$, otherwise the inequality $a_{2k} = b_{k+1} \leq a_{2k+1}$ leads to a contradiction. Thus by Clifford's theorem $k+1 \leq h^0(p'_1+...+p'_{2k}) \leq 1+\frac{2k}{2}$, its generic fibers is hyperelliptics unless $a_1 = a_2 = ... = a_{2g-2}$ which means $m_1 = ... = m_{2g-2} = 1$.

The hyperelliptic locus in a stratum $\Omega\mathcal{M}_g(m_1,...,m_k)$ induces from a stratum $\mathcal{Q}(d_1,...,d_k)$ satisfying $d_1+...+d_n = -4$. A singularity of order d_i of q give rise to two zeros of degree $m = d_i/2$ when d_i is even, single zero of degree $m = d+1$ when d is odd.

$$\sum_{d_j\texttt{odd}}(d_j+1) + \sum_{d_j\texttt{even}} d_j = 2g-2$$

By the formula of sums for the hyperelliptic locus in [8],

$$L(C) = \frac{1}{4}\sum_{d_j\texttt{odd}}\frac{1}{d_j+2} \leq \frac{1}{4}\sum_{d_j\texttt{odd}} 1 = \frac{g+1}{2}$$

a Teichmüller curve in the hyperelliptic locus satisfies $L(C) = \frac{g+1}{2}$ if and only if it is induced from $\mathcal{Q}(2k_1,...,2k_n,-1^{2g+2})$. $\square$

Remark 5.3. *D.W. Chen and M.Möller ([4]) have constructed a Teichmüller curve $C \in \Omega\mathcal{M}_3(1,1,1,1)$ with $L(C) = 2$, but it is not hyperelliptic: the square tiled surface given by the permutations*

$$(\pi_r = (1234)(5)(6789), \pi_\mu = (1)(2563)(4897))$$

They also have obtained a bound by using Cornalba-Harris-Xiao's slope inequality ([20]):

$$L(C) \leq \frac{3g}{(g-1)}\kappa_\mu = \frac{g}{4(g-1)}\sum_{i=1}^{k}\frac{m_i(m_i+2)}{m_i+1}$$

In fact we have obtained an upper bound of the slope of each graded quotient of the Harder-Narasimhan filtration of $f_*(\omega_{S/C})$ for Teichmüller curves:

Lemma 5.4. *For a Teichmüller curve which lies in $\Omega\mathcal{M}_g(m_1,...m_k)$, we have inequalities:*

$$w_i \leq 1 + a_{H_i(P)}$$

Here a_i is the i-th largest number in $\{-\frac{j}{m_i+1}|1 \leq j \leq m_i, 1 \leq i \leq k\}$, P is the special permutation (4) and $H_i(P) \geq 2i-2$.

Proof. For the vector bundle $f_*\mathcal{O}(m_1D_1+...+m_kD_k)$, the filtration (6) gives

$$0 \subset V'_1 \subset V'_2 ... \subset V'_g = f_*\mathcal{O}(m_1D_1+...+m_kD_k)$$

It is controlled by the following filtration:

$$0 \subset \mathcal{O} \subset \mathcal{O} \oplus V_{H_2(P)}/V_{H_2(P)-1} \subset ... \subset \mathcal{O} \oplus \bigoplus_{j=2}^{g} V_{H_j(P)}/V_{H_j(P)-1}$$

By lemma 2.1, $\mu_i(f_*\mathcal{O}(m_1D_1+...+m_kD_k)) \leq deg(V_{H_i(P)}/V_{H_i(P)-1})$. So we get

$$w_i = \mu_i(f_*(\omega_{S/C}))/deg(\mathcal{L}) = 1+\mu_i(f_*\mathcal{O}(m_1D_1+...+m_kD_k))/deg(\mathcal{L}) \leq 1+a_{H_i(P)}$$

$\square$

The Harder-Narasimhan filtration always give an upper bound of degrees of any sub vector bundles, especially those related to the sum of certain Lyapunov exponents.

Proposition 5.5. *If the VHS over the Teichmüller curve C contains a sub-VHS $\mathbb{W}$ of rank $2k$, then the sum of the k corresponding non-negative Lyapunov exponents is the sum of $w_{i_1}, ..., w_{i_k}$ (where i_j are different to each other) and satisfies*

$$\sum_{i=1}^{k} \lambda_i^{\mathbb{W}} \leq \sum_{i=1}^{k} (1 + a_{H_i(P)})$$

Proof. $\mathbb{W}^{(1,0)}$ is summand of $f_*(\omega_{S/C})$ by Deligne's semisimplicity theorem. The slope $\mu_j(\mathbb{W}^{(1,0)})$ is equal to $\mu_{i_j}(f_*(\omega_{S/C}))$ for some j by lemma 2.2, here we can choose i_j such that each other is different.

$$\sum_{i=1}^{k} \lambda_i^{\mathbb{W}} = \frac{2deg\mathbb{W}^{(1,0)}}{2g(C) - 2 + |\Delta|} = \frac{\sum_{j=1}^{k} \mu_j(\mathbb{W}^{(1,0)})}{deg(\mathcal{L})} = \sum_{j=1}^{k} \mu_{i_j}(f_*(\omega_{S/C}))/deg(\mathcal{L}) = \sum_{j=1}^{k} w_{i_j}$$

By lemma 5.4 and a_i decrease in i,

$$\sum_{i=1}^{k} \lambda_i^{\mathbb{W}} = \sum_{j=1}^{k} w_{i_j} \leq \sum_{i=1}^{k} (1 + a_{H_{i_j}(P)}) \leq \sum_{i=1}^{k} (1 + a_{H_i(P)})$$

□

We only present an example to explain the general principle on how to improve the upper bound when we know more information about Weierstrass semigroups of general fibers.

Corollary 5.6. *A Teichmüller curve which lies in the non hyperelliptic locus of $\mathcal{M}_4(2, 2, 1, 1)$ satisfies*

$$L(C) \leq 13/6$$

Proof. a_i equal: $-1/3, -1/3, -1/2, -1/2, -2/3, -2/3$. By Clifford theorem, $H_2(P) \geq 3, H_3(P) = 5, H_4(P) = 6$, so we choose the third (or the fourth), the fifth, the sixth element of $a_i : -1/2, -1/3, -1/3$. Finally we have

$$L(C) \leq \sum_{i=1}^{k} (1 + a_{H_i(P)}) = 13/6$$

□

This result has appeared in [4].

Proposition 5.7. *For a Teichmüller curve which satisfies the assumption 1.3 and lies in $\Omega\mathcal{M}_g(m_1, ...m_k)$, the i-th Lyapunov exponent satisfies the inequality::*

$$\lambda_i \leq 1 + a_{H_i(P)}$$

Here a_i is the i-th largest number in $\{-\frac{j}{m_i+1} | 1 \leq j \leq m_i, 1 \leq i \leq k\}$, P is the special permutation (4) and $H_i(P) \geq 2i - 2$.

Proof. The assumption 1.3 and the lemma 2.2 give us

$$grad(HN(f_*(\omega_{S/C}))) = (\bigoplus_{i=1}^{k} L_i) \oplus grad(HN(W))$$

so there are different j_i such that $\lambda_1 = w_{j_1} \geq \lambda_2 = w_{j_2} \geq ... \geq \lambda_k = w_{j_k}$. By lemma 5.4, we have

$$\lambda_i = w_{j_i} \leq w_i \leq 1 + a_{H_i(P)}$$

□

The equality can be reached for an algebraic primitive Teichmüller curve lying in the hyperelliptic locus induced from $\mathcal{Q}(2k_1, ..., 2k_n, -1^{2g+2})$.

6. Assumptions

Abelian covers. The Lyapunov spectrum has been computed for triangle groups ([2]), square tiled cyclic covers ([7] [11]) and square tiled abelian covers ([22]). They all satisfy the assumption 1.3. Here we give the description of square tiled cyclic covers:

Consider an integer $N \geq 1$ and a quadruple of integers (a_1, a_2, a_3, a_4) satisfying the following conditions:

$$0 < a_i \leq N; \quad gcd(N, a_1, ..., a_4) = 1; \quad \sum_{i=1}^{4} a_i \equiv 0 \quad (mod \quad N)$$

Let $z_1, z_2, z_3, z_4 \in \mathbb{C}$ be four distinct points. By $M_N(a_1, a_2, a_3, a_4)$ we denote the closed connected nonsingular Riemann surface obtained by normalization of the one defined by the equation

$$w^N = (z - z_1)^{a_1}(z - z_2)^{a_2}(z - z_3)^{a_3}(z - z_4)^{a_4}$$

Varying the cross-ratio (z_1, z_2, z_3, z_4) we obtain the moduli curve $\mathcal{M}_{(a_i),N}$. As an abstract curve it is isomorphic to $\mathcal{M}_{0,4} \simeq \mathbb{P}^1 - \{0, 1, \infty\}$; more strictly speaking, it should be considered as a stack. The canonical generator T of the group of deck transformations induces a linear map $T^* : H^{1,0}(X) \to H^{1,0}(X)$. $H^{1,0}(X)$ admits a splitting into a direct sum of eigenspaces $V^{1,0}(k)$ of T^* and satisfies the assumption 1.3. (cf. Theorem 2 in [7])

For even N, $M_N(N - 1, 1, N - 1, 1)$ has Lyapunov spectrum ([7]):

$$\{\frac{2}{N}, \frac{2}{N}, \frac{4}{N}, \frac{4}{N}, ..., \frac{N-2}{N}, \frac{N-2}{N}, 1\}$$

Remark 6.1. *By the Theorem 5.2 and the genus formular* $g = N+1-\frac{1}{2}\sum_{i=1}^{4} gcd(a_i, N)$, $M_N(N - 1, 1, N - 1, 1)$ *lies in the hyperelliptic locus which induced from* $\mathcal{Q}(N - 2, N - 2, -1^{2N})$, *because* $L(C)$ *equal* $\frac{g+1}{2}$.

Algebraic primitives. The variation of Hodge structures over a Teichmüller curve decomposes into sub-VHS

$$R^1 f_*\mathbb{C} = (\oplus_{i=1}^{r}\mathbb{L}_i) \oplus \mathbb{M} \tag{7}$$

Here $\mathbb{L}_i$ are rank-2 subsystems, maximal Higgs $\mathbb{L}_1^{1,0} \simeq \mathcal{L}$ for $i = 1$, non-unitary but not maximal Higgs for $i \neq 1$ ([18]). It is obvious that the Teichmüller curve satisfies the assumption 1.3 if $r \geq g - 1$.

If $r = g$, it is called algebraic primitive Teichmüller curves. We know there are only finite algebraic primitive Teichmüller curves in the stratum $\Omega\mathcal{M}_3(3, 1)$ by Möller and Bainbridge in [1], and they conjecture that the algebraic primitive Teichmüller curves in each stratum is finite ([20]).

Remark 6.2. *Algebraic primitive Teichmüller curves in the stratum* $\Omega\mathcal{M}_3(3, 1)$ *has Lyapunov spectrum* $\{1, \frac{2}{4}, \frac{1}{4}\}$ *by proposition 7.1.*

Wind-tree models. A wind-tree model or the infinite billiard table is defined as:

$$T(a,b) := \mathbb{R}^2 \setminus \bigcup_{m,n\in\mathbb{Z}} [m, m+a] \times [n, n+b]$$

with $0 < a, b < 1$. Denote by $\phi_t^\theta : T(a,b) \to T(a,b)$ the billiard flow: for a point $p \in T(a,b)$, the point ϕ_t^θ is the position of a particle after time t starting from position p in direction θ.

Theorem 6.3. *([6])Let $d(.,.)$ be the Euclidean distance on $\mathbb{R}^2$.*

- *(Case 1) If a and b are rational numbers or can be written as $1/(1-a) = x + y\sqrt{D}, 1/(1-b) = (1-x) + y\sqrt{D}$ with $x, y \in \mathbb{Q}$ and D a positive square-free integer then for Lebesgue almost all θ and every point p in $T(a,b)$.*
- *(Case 2) For Lebesgue-almost all $(a,b) \in (0,1)^2$, Lebesgue-almost all θ and every point p in $T(a,b)$ (with an infinite forward orbit):*

$$\limsup_{T\to+\infty} \frac{logd(p, \phi_T^\theta(p))}{logT} = \frac{2}{3} \tag{8}$$

We are interested in the case 1 because it is related to Teichmüller curves. By the Katok-Zemliakov construction, the billiard flow can be replaced by a linear flow on a(non compact)translation surface which is made of four copies of $T(a,b)$ that we denote $X_\infty(a,b)$. The surface $X_\infty(a,b)$ is $\mathbb{Z}^2$ -periodic and we denote by $X(a,b)$ the quotient of $X_\infty(a,b)$ under the $\mathbb{Z}^2$ action.

The surface $X(a,b)$ is a covering (with Deck group $\mathbb{Z}/2 \times \mathbb{Z}/2$) of the genus 2 surface $L(a,b) \in \Omega\mathcal{M}_2(2)$ which is called L-shaped surface ([3] [16]). The orbit of $X(a,b)$ for the Teichmüller flow belongs to the moduli space $\Omega\mathcal{M}_5(2,2,2,2)$.

The Teichmüller curve generated by the surface $X(a,b)$ satisfies the assumption 1.3 because there is an $SL_2(\mathbb{R})$-equivalent splitting of the Hodge bundle. Its Lyapunov spectrum is $\{1, \frac{2}{3}, \frac{2}{3}, \frac{1}{3}, \frac{1}{3}\}$, the equation (8) is equivalence to say that $\lambda_2 = \frac{2}{3}$ ([6]).

Remark 6.4. *In fact, a Teichmüller curve which satisfies the assumption 1.3 and lies in $\Omega\mathcal{M}_5(2,2,2,2)$ satisfies $\lambda_2 \le \frac{2}{3}$ by the proposition 5.7. By the Theorem 5.2, $X(a,b)$ is lies in the hyperelliptic locus which induced from $\mathcal{Q}(4,4,-1^{12})$, because $L(C)$ equal $\frac{g+1}{2}$.*

7. Non varying strata

Recently, there are many progresses about the phenomenon that the sum of Lyapunov exponents is non varying in some strata ([4][5][23]). The following proposition is a an immediate corollary of the theorem 3.5.

Proposition 7.1. *For a Teichmüller curve which satisfies the assumption 1.3 and lies in hyperelliptic loci or one of the following strata:*
$\overline{\Omega\mathcal{M}}_3(4), \overline{\Omega\mathcal{M}}_3(3,1), \overline{\Omega\mathcal{M}}_3^{odd}(2,2), \overline{\Omega\mathcal{M}}_3(2,1,1)$
$\overline{\Omega\mathcal{M}}_4(6), \overline{\Omega\mathcal{M}}_4(5,1), \overline{\Omega\mathcal{M}}_4^{odd}(4,2), \overline{\Omega\mathcal{M}}_4^{non-hyp}(3,3), \overline{\Omega\mathcal{M}}_4^{odd}(2,2,2), \overline{\Omega\mathcal{M}}_4(3,2,1)$
$\overline{\Omega\mathcal{M}}_5(8), \overline{\Omega\mathcal{M}}_5(5,3), \overline{\Omega\mathcal{M}}_5^{odd}(6,2)$
The i-th Lyapunov exponent λ_i equals the w_i which is computed in the theorem 3.5.

Proof. The assumption 1.3 and the lemma 2.2 give us

$$grad(HN(f_*(\omega_{S/C}))) = (\bigoplus_{i=1}^{k} L_i) \oplus grad(HN(W))$$

We have constructed the Harder-Narasimhan filtration with $w_i > 0$ in [23]. If $k < g$, then $deg(W) = 0$ by the assumption 1.3. Using lemma 2.2, we get

$$0 = \frac{deg(W)}{deg(\mathcal{L})} = \frac{\sum_{i=1}^{g-k} \mu_i(W)}{deg(\mathcal{L})} = \frac{\sum_{i=1}^{g-k} \mu_{j_i}(f_*(\omega_{S/C}))}{deg(\mathcal{L})} = \sum_{i=1}^{g-k} w_{j_i} > 0$$

It is contradiction! Thus we have $grad(HN(f_*(\omega_{S/C}))) = \bigoplus_{i=1}^{g} L_i$ and $\lambda_i = w_i$. $\square$

Hyperelliptic loci. It has been shown in [7] that the "stairs" square tiled surface $S(N)$ satisfies the assumption 1.3 and belongs to the hyperelliptic connected component $\overline{\Omega\mathcal{M}}_g^{hyp}(2g-2)$, for $N = 2g-1$ or $\overline{\Omega\mathcal{M}}_g^{hyp}(g-1, g-1)$, for $N = 2g$.

Remark 7.2. *The Proposition 7.1 also implies that the Lyapunov spetrum of the Hodge bundles over the corresponding arithmetic Teichmüller curves is*

$$\Lambda Spec = \begin{cases} \frac{1}{N}, \frac{3}{N}, \frac{5}{N}, ..., \frac{N}{N} & N = 2g-1 \\ \frac{2}{N}, \frac{4}{N}, \frac{6}{N}, ..., \frac{N}{N} & N = 2g \end{cases}$$

Which has been shown in [7] *by using the fact $S(N)$ is quotient of $M_N(N-1, 1, N-1, 1)$ (resp. $M_{2N}(2N-1, 1, N, N)$) for N is even (resp. odd).*

Prym varieties. McMullen, use Prym eigenforms, has constructed infinitely many primitive Teichmüller curves for $g = 2, 3$ and 4 ([17]). Let $W_D(6)$ be the Prym Teichmüller curves in $\Omega\mathcal{M}_4$. It has VHS decomposition:

$$R^1 f_* \mathbb{C} = (\mathbb{L}_1 \oplus \mathbb{L}_2) \oplus \mathbb{M}$$

So it map to curves W_D^X in the Hilbert modular surface $X_D = \mathbb{H}^2/SL(\mathcal{O}_D \oplus \mathcal{O}_D^\vee)$.

Remark 7.3. *The proposition 5.5 tells us that the number $deg(\mathbb{L}_2^{1,0})/deg(\mathcal{L})$ equals one of the numbers $\{\frac{4}{7}, \frac{2}{7}, \frac{1}{7}\}$. In fact it has been shown that W_D^X is the vanishing locus of a modular form of weight $(2, 14)$, so $deg(\mathbb{L}_2^{1,0})/deg(\mathcal{L})$ is $\frac{1}{7}$. (*[20][21]*)*

References

[1] M.Bainbridge and M.Möller, *The Deligne-Mumford compactification of the real multiplication locus and Teichmüller curves in genus three.* Acta Mathematica 208(1): 1-92, 2012

[2] I.Bouw and M.Möller, *Teichmüller cuves, triangle groups and Lyapunov exponents.* Ann of math.(2) 172(1):139-185, 2010

[3] K.Calta, *Veech surfaces and complete periodicity in genus two* J. Amer. Math. 228 (2011), Soc. 17 (4)(2004), 871-908

[4] D.W.Chen and M.Möller, *Non-varying sum of Lyapunov exponents of Abelian differetials in low genus* arxiv.org/abs/1104.3932

[5] D.W.Chen and M.Möller, *Quadratic differentials in low genus: exceptional and non-varying* arxiv.org/abs/1204.1707

[6] V.Delecroix, P.Hubert and S.Leliévre, *Diffusion for the periodic wind-tree model.* arxiv.org/abs/1107.1810

[7] A.Eskin, M.Kontsevich and A.Zorich,*Lyapunov spectrum of square-tiled cyclic covers* . J. Mod. Dyn. 5 (2011), no. 2, 319-353,

[8] A.Eskin, M.Kontsevich and A.Zorich, *Sum of Lyapunov exponents of the Hodge bundle with respect to the Teichmüller geodesic flow.* arxiv.org/abs/1112.5872

[9] A.Eskin, H.Masur and A.Zorich, *Moduli spaces of Abelian differentials:the principal boundary,counting problems and the Siegel-Veech constats.* Publ. Math. Inst. Hautes Etudes, 97, 61-179, 2003

[10] Giovanni Forni, *Deviation of ergodic averages for area-preserving ows on surfaces of higher genus.* Ann. of Math. (2) 155 (2002), no. 1, 1-103.

[11] Giovanni Forni, Carlos Matheus and Anton Zorich, *Square-tiled cyclic covers*. J. Mod. Dyn. 5 (2011), no.2, 285-318

[12] D.Huybrechts and M.Lehn, *The Geometry of Moduli spaces of sheaves*. Aspects of Mathematics 31, Friedr. Vieweg and Sohn, Braunschweig (1997).

[13] R.Hartshorne, *Algebraic geometry*. GTM 52 (1977) Springer, New York.

[14] M.Kontsevich and A.Zorich, *Lyapunov exponents and Hodge theory*. arxiv.org/abs/hep-th/9701164

[15] M.Kontsevich and A.Zorich, *Connected components of the moduli spaces of Abelian differentials with prescribed singularities*. Invent. Math. 153(3): 631-678, 2003.

[16] C.T.McMullen, *Billiards and Teichmüller curves on Hilbert modular surfaces*. J. Amer. Math. Soc. 16 (2003), no. 4, 857-885

[17] C.T.McMullen, *Prym varieties and Teichmüller curves*. Duke Math.J. 133(2006), no. 3, 569-590.

[18] M.Möller, *Shimura and Teichmüller curves*. J. Mod. Dyn. 5 (2011), no. 1, 1-32

[19] M.Möller, *Prym covers, theta functions and Kobayashi curves in Hilbert modular surfaces*. arxiv.org/abs/1111.2624

[20] M.Möller,*Teichmüller curves, mainly from the view point of algebraic geometry*. available on the author's webpage, to appear as PCMI lecture notes.

[21] C.Weiß, *Twisted Teichmüller curves*. arxiv.org/abs/1208.1895

[22] A.Wright, *Schwarz triangle mappings and Teichmller curves: abelian square-tiled surfaces*. arxiv.org/abs/1203.2683

[23] F.Yu and K.Zuo, *Weierstrass filtration on Teichmüller curves and Lyapunov exponents*. arxiv.org/abs/1203.6053

[24] A.Zorich, *Flat surfaces*. In Frontiers in Number Theory, Physics and Geometry. Volume 1:On random matrices, zeta functions and dynamical systems, pages 439-586. Springer-Verlag, Berlin, 2006.

Universität Mainz, Fachbereich 17, Mathematik, 55099 Mainz, Germany
E-mail address: `yuf@uni-mainz.de,vvyufei@gmail.com`

Universität Mainz, Fachbereich 17, Mathematik, 55099 Mainz, Germany
E-mail address: `zuok@uni-mainz.de`